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Normal Forms of Vector Fields and Diffeomorphisms

By

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Abstract

We shall show simultaneous normal forms of a system of vector fields and local diffeomorphisms under Brjuno condition. These results are proved by a new scheme of a generalized implicit function theorem with high loss of derivatives such that for some $\varepsilon, 0 < \varepsilon < 1$,

$$\exp(e^{\frac{1}{(\sigma-\sigma')^{\varepsilon}}}), \qquad 0 < \sigma' < \sigma.$$

The nonlinear equations (homological equation) are overdetermined system of equations.

1 Normal forms of vector fields

Let us consider a system of analytic vector fields $X^\mu(\mu = 1, \cdots, d)$ in some neighborhood of the origin of $x = (x_1, \cdots, x_n) \in \boldsymbol{R}^n$,

$$X^\mu = \langle X^\mu, \partial_x \rangle = \sum_{j=1}^{n} X_j^\mu(x)\partial_{x_j} \quad 1 \le \mu \le d \tag{1}$$

$$\partial_x = (\partial_{x_1}, \cdots, \partial_{x_n}), \quad \partial_{x_j} = \frac{\partial}{\partial x_j}.$$

We assume

$$X^\mu \ (1 \le \mu \le d) \ \text{ are singular i.e. } \ X^\mu(0) = 0 \ \text{ for } \ 1 \le \mu \le d. \tag{2}$$

The linear parts of $X^\mu(1 \le \mu \le d)$ are semi-simple i.e.,

$$X^\mu(x) = (X_1^\mu(x), \cdots, X_n^\mu(x)) = \Lambda^\mu x + R^\mu(x), \quad 1 \le \mu \le d, \tag{3}$$

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where

$$\Lambda^\mu = \begin{pmatrix} \lambda_1^\mu & & 0 \\ & \ddots & \\ 0 & & \lambda_n^\mu \end{pmatrix}, \qquad \lambda_j^\mu \in \boldsymbol{C}$$

and where $R^\mu(x)$ are analytic at the origin and satisfy

$$R^\mu(0) = \partial_x R^\mu(0) = 0, \quad 1 \leq \mu \leq d.$$

(4) $\quad X^\mu (1 \leq \mu \leq d)$ are pairwise commuting, i. e. $[X^\mu, X^\nu] = 0, \quad 1 \leq \nu, \mu \leq d.$

Set $\lambda^\mu = (\lambda_1^\mu, \cdots, \lambda_n^\mu), \quad (1 \leq \mu \leq d)$. We are interested in reduction of vector fields to normal forms. If $d = 1$ (single case), a normal form was obtained by Poincaré under the condition

$$(*) \qquad |\lambda\alpha| \geq c_0|\alpha| \quad \text{for } \alpha \in \boldsymbol{Z}_+^n, \ |\alpha| \gg 1$$

Roughly speaking, in order to find a change of variables which reduces a vector field to its normal form we are to solve a nonlinear equation, a so-called homological equation. The condition (*) assures the bounded inverse of a linearized operator.
As to the solvability of nonlinear equations including homological equations under Poincaré condition there is a generalization by Kaplan.([6]).

The solvability of these nonlinear equations with unbounded inverse was proved by Siegel in case $d = 1$. ([12]) under a famous Siegel condition :

$$(5) \qquad \exists c > 0, \exists \gamma > 0; |\lambda\alpha - \lambda_k| \geq c|\alpha|^{-\gamma} \quad \text{for } 1 \leq k \leq n, \ \alpha \in \boldsymbol{Z}_+^n.$$

Rüssman ([10]) generalized his idea and proved

Assume $d = 1$. Suppose (2), (3) and (5). Then the vector field (1) can be transformed to a normal form by a holomorphic change of variables.

By recent study of normal forms of mappings by Yoccoz ([13]) and M. Perez ([9]), it is natural to weaken the condition (5) to the following Brjuno condition:

$\exists c > 0, \exists \gamma > 0$ such that

$$(6) \qquad \max_{1 \leq \mu \leq d} |\lambda^\mu \alpha - \lambda_k^\mu| \geq c \exp\left(-\frac{|\alpha|}{\log(2 + |\alpha|)^{1+\gamma}}\right) \quad \forall \alpha \in \boldsymbol{Z}_+^n, \ 1 \leq \forall k \leq n.$$

In this note, we are also interested in the solvability of overdetermined systems. We shall prove

Theorem
Let $X^1(x), \cdots, X^d(x)$ be pairwise commuting analytic singular vector fields satisfying (2)

and (3). If λ^μ $(1 \le \mu \le d)$ verify the Brjuno condition (6) we can find a neighborhood of the origin and an analytic change of variables $x = y + u(y)$ which transforms simultaneously $X^j(x)(1 \le j \le d)$ into

$$\lambda_1^j y_1 \partial_{y_1} + \cdots + \lambda_n^j y_n \partial_{y_n}, \quad 1 \le j \le d,$$

i.e. X^μ is linearizable.

Sketch of the proof. The existence of the diffeomorphism u is equivalent to solving the equation $(1 + \partial u/\partial x)^{-1} X^\mu(x + u(x)) = \Lambda^\mu x$. If we define a homological operator $\mathcal{L}_{\lambda^\mu}$ ($1 \le \mu \le d$) by

$$\mathcal{L}_{\lambda^\mu} = \operatorname{diag}(\Lambda^\mu x \partial_x - \lambda_1^\mu, \cdots, \Lambda^\mu x \partial_x - \lambda_n^\mu) \quad \Lambda^\mu x \partial_x = \sum_{j=1}^n \lambda_j^\mu x_j \partial_j, \tag{7}$$

the equation is equivalent to solving the following system of homological equations

$$\mathcal{L}_{\lambda^\mu} u - R^\mu(y + u) = 0, \ 1 \le \mu \le d. \tag{8}$$

We define

$$H(T) = \{u(x) = \sum_{\alpha \in \mathbf{Z}^n} u_\alpha x^\alpha \in \mathcal{O}(\Omega) : \|u\|_T = \sum_{\alpha \in \mathbf{Z}^n} |u_\alpha| T^{|\alpha|} < \infty\},$$

with $\mathcal{O}(\Omega)$ being the set of holomorphic functions in Ω, and

$$Mf := \sum_{\mu=1}^d \mathcal{L}_{\bar{\lambda}_\mu} \mathcal{L}_{\lambda_\nu} f, \quad f \in H(T) \times \cdots \times H(T) = H(T)^n, \quad T > 0.$$

Clearly, $H(T)$ is a scale of Banach spaces. Moreover, we have the following

Lemma *Suppose the Brjuno condition (6). Then, $0 < \forall T' < \forall T$, there exists $\exists M^{-1}$: $H(T')^n \to H(T)^n$ such that*

$$\|M^{-1}\|_{T' \to T} \le \exp\left(2 \exp\left(\frac{c}{(T - T')^{\frac{1}{1+\tau}}}\right)\right).$$

Proof of Theorem.
Suppose $v_0, \cdots, v_{k-1}$ is defined. After a change of variables $(1 + \partial_x v_0) \circ \cdots \circ (1 + \partial_x v_{k-1})$, let us suppose that the vector fields X^μ be transformed into $X^{\mu,k} = \Lambda^\mu x + R_\mu^k(x)$. We define

$$v_k = \sum_{\nu=1}^k \mathcal{L}_{\overline{\lambda_\nu}} M^{-1} R_\nu^k, \qquad X^{\mu,k+1} = (1 + \partial_x v_k)^{-1} X^{\mu,k}(x + v_k(x)). \tag{9}$$

Implicit function theorem with high loss of derivatives shows that the change of variables $(1+\partial_x v_0)\circ\cdots\circ(1+\partial_x v_k)\circ\cdots$ gives the desired one. $\square$

If the condition (3) (semi-simple) is not true, it is known examples ($d=1$) that the above diffeomorphism does not exist under a Siegel condition in general.(cf. [4]). In view of this we consider a system of commuting vector fields

$$X^\mu = J^\mu x + R^\mu(x), \quad \mu = 1,\ldots,d, \tag{10}$$

where we may assume that J^μ is in a Jordan normal form and $R^\mu(0)=\partial R^\mu(0)=0$. We assume the following Strong Siegel condition

$$\exists c_0>0;\ \max_{1\le\mu\le d}|\lambda^\mu d-\lambda_k^\mu|\ge c_0 \quad \text{for all } \alpha\in \boldsymbol{Z}_+^n,\ 1\le\forall k\le n. \tag{11}$$

Then we have

Theorem *Let $X^j(x)(j=1,\cdots,d)$ be pairwise commuting holomorphic vector fields as above. Assume the strong Siegel condition (11). Then the vector fields (11) are simultaneously transformed into their linear parts by an analytic change of variables.*

2 Normal forms of holomorphic diffeomorphisms

Let $\Phi:\boldsymbol{C}^n\to\boldsymbol{C}^n$ be an analytic diffeomorphisms such that $\Phi(0)=0$. Suppose that

$$\Phi(x)=\Lambda x+\varphi(x), \quad \Lambda;\text{diagonal},\ deg\varphi\ge 2.$$

Then, if the modulus of all eigenvalues of Λ are greater than 1 or smaller than 1 then we can find a diffeomorphism $\varphi;\boldsymbol{C}^n\to\boldsymbol{C}^n$, $\varphi(0)=0$ such that $\varphi^{-1}\circ\Phi\circ\varphi=\Lambda x$. (Poincaré). The case of modulus $=1$ was proved by Siegel ([11]) if $n=1$ under Siegel condition. Recently, J. Moser ([8]) has extended this result to the following situation ($n=1,\ d\ge 1$) : Consider a system of analytic diffeomorphisms $\Phi_\nu:\boldsymbol{C}\to\boldsymbol{C}$ $(1\le\nu\le d)$ such that $\Phi_\nu(0)=0$ $(1\le\nu\le d)$. Assume that Φ_ν are mutually commuting, i.e.,

$$\Phi_\nu\circ\Phi_\mu=\Phi_\mu\circ\Phi_\nu \quad (\nu,\mu=1,\cdots d).$$

Suppose that

$$\Phi_\nu(x)=\Lambda^\nu x+\varphi_\nu(x), \quad \nu=1,\cdots d,\ \Lambda^\nu:\text{diagonal}, \quad deg\varphi_\nu\ge 2$$

Then he proved :

Suppose the Siegel condition (5). Then there exists a local diffeomorphism φ $\varphi; \boldsymbol{C} \to \boldsymbol{C}$, $\varphi(0) = 0$ *such that*

$$\varphi^{-1} \circ \Phi_\nu \circ \varphi = \Lambda_\nu x, \qquad (1 \leq \nu \leq d).$$

We can prove the following theorem.

Theorem *Suppose the Brjuno condition (6). Then there exists a local diffeomorphism* $\varphi : \boldsymbol{C}^n \to \boldsymbol{C}^n, \varphi(0) = 0$ *such that*

$$\varphi^{-1} \circ \Phi_\nu \circ \varphi = \Lambda^\nu x, \qquad (1 \leq \nu \leq d).$$

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